Qualitative properties of solutions for an integral system related to the Hardy–Sobolev inequality

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Abstract

This article carries out a qualitative analysis on a system of integral equations of the Hardy–Sobolev type. Namely, results concerning Liouville type properties and the fast and slow decay rates of positive solutions for the system are established. For a bounded and decaying positive solution, it is shown that it either decays with the slow rates or the fast rates depending on its integrability. Particularly, a criterion for distinguishing integrable solutions from other bounded and decaying solutions in terms of their asymptotic behavior is provided. Moreover, related results on the optimal integrability, boundedness, radial symmetry and monotonicity of positive integrable solutions are also established. As a result of the equivalence between the integral system and a system of poly-harmonic equations under appropriate conditions, the results translate over to the corresponding poly-harmonic system. Hence, several classical results on semilinear elliptic systems are recovered and further generalized.

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1 Introduction and the main results

In this paper, we study the qualitative properties of positive solutions for an integral system of the Hardy–Sobolev type. In particular, we consider the

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system of integral equations involving Riesz potentials and Hardy terms,

$$
\begin{cases}
u(x) = \displaystyle\int_{\mathbb{R}^n} \frac{v(y)^q}{|x-y|^{n-\alpha}|y|^{\sigma_1}}\, dy, \\[4mm]
v(x) = \displaystyle\int_{\mathbb{R}^n} \frac{u(y)^p}{|x-y|^{n-\alpha}|y|^{\sigma_2}}\, dy,
\end{cases}
\tag{1.1}
$$

where throughout we assume that $n \geq 3$, $p, q > 0$ with $pq > 1$, $\alpha \in (0, n)$ and $\sigma_1, \sigma_2 \in [0, \alpha)$. As a result, we establish analogous properties for the closely related system of semilinear differential equations with singular weights,

$$
\begin{cases}
(-\Delta)^{\alpha/2} u(x) = \dfrac{v(x)^q}{|x|^{\sigma_1}} & \text{in } \mathbb{R}^n \backslash \{0\}, \\[4mm]
(-\Delta)^{\alpha/2} v(x) = \dfrac{u(x)^p}{|x|^{\sigma_2}} & \text{in } \mathbb{R}^n \backslash \{0\},
\end{cases}
\tag{1.2}
$$

since both systems are equivalent under appropriate conditions. Particularly, if $p, q > 1$ and $\alpha = 2k$ is an even positive integer, then a positive classical solution $u, v \in C^{2k}(\mathbb{R}^n \backslash \{0\}) \cap C(\mathbb{R}^n)$ of system (1.1), multiplied by suitable constants if necessary, satisfies the poly-harmonic system (1.2) pointwise except at the origin; and vice versa (cf. [8, 40, 41]).

Our aim is to fully characterize the positive solutions, specifically the ground states, in terms of their asymptotic behavior and elucidate its connection with Liouville type non-existence results. The motivation for studying these properties for the Hardy–Sobolev type systems arises from several related and well-known problems. For example, one problem originates from the doubly weighted Hardy–Littlewood–Sobolev (HLS) inequality [38], which states that for $r, s \in (1, \infty)$, $\alpha \in (0, n)$ and $0 \leq \sigma_1 + \sigma_2 \leq \alpha$,

$$
\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^{\sigma_1}|x-y|^{n-\alpha}|y|^{\sigma_2}}\, dxdy \right| \leq C_{\sigma_i, s, \alpha, n} \|f\|_{L^r(\mathbb{R}^n)} \|g\|_{L^s(\mathbb{R}^n)},
$$

where

$$
\frac{\alpha}{n} - \frac{1}{r} < \frac{\sigma_1}{n} < 1 - \frac{1}{r} \quad \text{and} \quad \frac{1}{r} + \frac{1}{s} + \frac{\sigma_1 + \sigma_2}{n} = \frac{n+\alpha}{n}.
$$

Here and throughout this paper, $\|f\|_{L^p(\mathbb{R}^n)}$ or $\|f\|_p$ denotes the norm of f in the Lebesgue space $L^p(\mathbb{R}^n)$. To find the best constant in the doubly weighted HLS inequality, we maximize the functional

$$
J(f, g) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^{\sigma_1}|x-y|^{n-\alpha}|y|^{\sigma_2}}\, dxdy
\tag{1.3}
$$

under non-negative functions f and g with the constraints

$$\|f\|_{L^r(\mathbb{R}^n)} = \|g\|_{L^s(\mathbb{R}^n)} = 1.$$

Setting $u = c_1 f^{r-1}$ and $v = c_2 g^{s-1}$ with proper choices for the constants c_1 and c_2 and taking $\frac{1}{p+1} = 1 - \frac{1}{r}$ and $\frac{1}{q+1} = 1 - \frac{1}{s}$ with $pq \neq 1$, the corresponding Euler–Lagrange equations for the extremal functions of the functional are equivalent to the so-called weighted HLS integral system

$$\begin{cases} u(x) = \dfrac{1}{|x|^{\sigma_1}} \displaystyle\int_{\mathbb{R}^n} \dfrac{v(y)^q}{|x-y|^{n-\alpha}|y|^{\sigma_2}}\, dy, \\[4mm] v(x) = \dfrac{1}{|x|^{\sigma_2}} \displaystyle\int_{\mathbb{R}^n} \dfrac{u(y)^p}{|x-y|^{n-\alpha}|y|^{\sigma_1}}\, dy, \end{cases} \tag{1.4}$$

where

$$\frac{\sigma_1}{n} < \frac{1}{p+1} < \frac{n-\alpha+\sigma_1}{n} \ \text{ and } \ \frac{1}{1+q} + \frac{1}{1+p} = \frac{n-\alpha+\sigma_1+\sigma_2}{n}.$$

Notice that (1.1) and (1.4) coincide if $\sigma_1 = \sigma_2 = 0$. Now when $\sigma_1 = \sigma_2 = 0$ and $r = s = \frac{2n}{n+\alpha}$, Lieb classified all maximizers of the functional (1.3) and posed the classification of all the critical points as an open problem in [29], which was later settled by Chen, Li and Ou in [11].

If we take $\sigma_1 = \sigma_2 \doteq \sigma$, $p = q$ and $u \equiv v$, system (1.1) becomes the integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{u(y)^p}{|x-y|^{n-\alpha}|y|^{\sigma}}\, dy. \tag{1.5}$$

In the special case where $\alpha = 2$ and $p = \frac{n+\alpha-2\sigma}{n-\alpha}$, (1.5) is closely related to the Euler–Lagrange equation for the extremal functions of the classical Hardy–Sobolev inequality, which states there exists a constant C for which

$$\left(\int_{\mathbb{R}^n} \frac{u(x)^{\frac{2(n-\sigma)}{n-2}}}{|x|^{\sigma}} \right)^{\frac{n-2}{n-\sigma}} \leq C \int_{\mathbb{R}^n} |\nabla u(x)|^2\, dx \ \text{ for all } \ u \in \mathcal{D}^{1,2}(\mathbb{R}^n).$$

In fact, the Hardy–Sobolev inequality is a special case of the Caffarelli–Kohn–Nirenberg inequality (cf. [2, 3, 5, 12]). Furthermore, the classification of solutions for the unweighted integral equation and its corresponding differential equation provide an important ingredient in the Yamabe and prescribing scalar curvature problems.

Another noteworthy and related issue concerns Liouville type theorems. Such non-existence results are important in deriving singularity estimates

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and a priori bounds for solutions of Dirichlet problems for a class of elliptic equations (cf. [17, 35]). The Hénon–Lane–Emden system, which coincides with (1.2) when $\alpha = 2$ and $\sigma_1, \sigma_2 \in (-\infty, 2)$, has garnered some recent attention with respect to the Hénon–Lane–Emden conjecture, which states that the system admits no positive classical solution in the subcritical case

$$\frac{n - \sigma_1}{1 + q} + \frac{n - \sigma_2}{1 + p} > n - 2.$$

In [34], Phan proved the conjecture for $n = 3$ in the class of bounded solutions and for $n = 3, 4$ provided that $\sigma_1, \sigma_2 \geq 0$ (see also [16]). For the higher dimensional case or the general poly-harmonic version, this conjecture has partial results (cf. [15, 34, 40] and the references therein). In [39], the author verified that the poly-harmonic version is indeed sharp by establishing the existence of positive solutions for (1.2) in the non-subcritical case (see also [24, 30]). Even in the unweighted case (i.e. $\sigma_1 = \sigma_2 = 0$) with $\alpha = 2$, the conjecture, more commonly known as the Lane–Emden conjecture, still has only partial results. Specifically, it is true for radial solutions [33] and for $n \leq 4$ [35, 36, 37] (cf. [1, 31] for the poly-harmonic case).

In [26] and [27], the authors examined the decay properties of positive solutions for the Lane–Emden equation,

$$-\Delta u(x) = u(x)^p \ \text{ in } \ \mathbb{R}^n.$$

It was shown that solutions decay to zero at infinity with either the fast rate $u(x) \simeq |x|^{-(n-2)}$ or with the slow rate $u(x) \simeq |x|^{-\frac{2}{p-1}}$, where the notation $f(x) \simeq g(x)$ means there exist positive constants c and C such that $cg(x) \leq f(x) \leq Cg(x)$ as $|x| \longrightarrow \infty$. Analogous studies on the asymptotic properties of solutions for the weighted integral equation (1.5) can be found in [19] and for the unweighted version of system (1.1) in [20]. In addition, results on the regularity of solutions for these equations and systems can be found in various papers (cf. [6, 10, 14, 28, 32]).

We are now ready to describe the main results of this paper. Henceforth, we define

$$p_0 \doteq \frac{\alpha(1 + p) - (\sigma_2 + \sigma_1 p)}{pq - 1} \ \text{ and } \ q_0 \doteq \frac{\alpha(1 + q) - (\sigma_1 + \sigma_2 q)}{pq - 1}.$$

We also define some notions of solutions for the Hardy–Sobolev type system including the integrable solutions.

Definition. *Let u, v be positive solutions of* (1.1). *Then u, v are said to be:*

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(i) ***decaying*** *solutions if* $u(x) \simeq |x|^{-\theta_1}$ *and* $v(x) \simeq |x|^{-\theta_2}$ *for some rates* $\theta_1, \theta_2 > 0$;

(ii) ***integrable*** *solutions if* $u \in L^{r_0}(\mathbb{R}^n)$ *and* $v \in L^{s_0}(\mathbb{R}^n)$ *where*

$$r_0 = \frac{n}{q_0} \quad and \quad s_0 = \frac{n}{p_0}.$$

Definition. *Let* u, v *be positive solutions of* (1.1). *Then* u, v *are said to decay with the **slow rates** as* $|x| \longrightarrow \infty$ *if* $u(x) \simeq |x|^{-q_0}$ *and* $v(x) \simeq |x|^{-p_0}$. *Suppose* $q \geq p$ *and* $\sigma_1 \geq \sigma_2$. *Then* u, v *are said to decay with the **fast rates** as* $|x| \longrightarrow \infty$ *if*

$$u(x) \simeq |x|^{-(n-\alpha)}$$

and

$$\begin{cases} v(x) \simeq |x|^{-(n-\alpha)}, & if\ p(n-\alpha) + \sigma_2 > n; \\ v(x) \simeq |x|^{-(n-\alpha)} \ln |x|, & if\ p(n-\alpha) + \sigma_2 = n; \\ v(x) \simeq |x|^{-(p(n-\alpha)-(\alpha-\sigma_2))}, & if\ p(n-\alpha) + \sigma_2 < n. \end{cases}$$

Remark 1. *The conditions on the parameters in the previous definition, i.e.* $q \geq p$ *and* $\sigma_1 \geq \sigma_2$, *which we will sometimes assume within our main theorems, are not so essential. Namely, the results still remain true if we interchange the parameters provided that* u *and* v *are interchanged accordingly in the definition and the theorems.*

Remark 2. *In view of the equivalence with poly-harmonic systems and the regularity theory indicated by the earlier references, **classical solutions** of* (1.1) *should be understood to mean solutions belonging to* $C^{\lfloor \alpha \rfloor}(\mathbb{R}^n \backslash \{0\}) \cap C(\mathbb{R}^n)$, *where* $\lfloor \cdot \rfloor$ *is the greatest integer function.*

1.1 Main results

Theorem 1. *There hold the following.*

(i) *If* u, v *are bounded and decaying solutions of* (1.1), *then there exists a positive constant* C *such that as* $|x| \longrightarrow \infty$,

$$u(x) \leq C|x|^{-q_0} \quad and \quad v(x) \leq C|x|^{-p_0}.$$

(ii) *Suppose* $q \geq p$ *and* $\sigma_1 \geq \sigma_2$ *(so that* $q_0 \geq p_0$*), and let* u, v *be positive solutions of* (1.1). *Then there exists a positive constant* c *such that as* $|x| \longrightarrow \infty$,

$$u(x) \geq \frac{c}{|x|^{n-\alpha}} \quad and \quad v(x) \geq \frac{c}{|x|^{\min\{n-\alpha, p(n-\alpha)-(\alpha-\sigma_2)\}}}.$$

We now introduce our Liouville type theorem for the Hardy–Sobolev type system. Basically, our result states that the system has no non-negative ground state classical solutions in the subcritical case besides the trivial pair $u, v \equiv 0$.

Theorem 2. *Suppose that, in addition, $\alpha \in (1, n)$. Then system (1.1) does not admit any bounded and decaying positive classical solution whenever the following subcritical condition holds*

$$\frac{n - \sigma_1}{1 + q} + \frac{n - \sigma_2}{1 + p} > n - \alpha. \tag{1.6}$$

Remark 3. *In [40], the author proved the following non-existence theorem but without any boundedness or decaying assumption on positive solutions.*

Theorem A. *System (1.1) has no positive solution if either $pq \in (0, 1]$ or when $pq > 1$ and $\max\{p_0, q_0\} \geq n - \alpha$.*

Interestingly, Theorem A and Theorem 2 are reminiscent of the non-existence results of Serrin and Zou [36] for the Lane–Emden system. We should also mention the earlier work in [13], which also obtained similar Liouville theorems among other interesting and related results.

Theorem 2 applies to integrable solutions as well. In particular, we later show that integrable solutions are indeed radially symmetric and decreasing about the origin. Therefore, the proof of Theorem 2 can be adopted in this situation to get the following.

Corollary 1. *Suppose that, in addition, $\alpha \in (1, n)$. Then system (1.1) does not admit any positive integrable solution whenever the subcritical condition (1.6) holds.*

The next theorem concerns the properties of integrable solutions. More precisely, it states that we can distinguish integrable solutions from the other ground states with "slower" decay rates in that integrable solutions are equivalently characterized as the bounded positive solutions which decay with the fast rates. In dealing with integrable solutions, Corollary 1 indicates that we can restrict our attention to the non-subcritical case:

$$q_0 + p_0 \leq n - \alpha, \tag{1.7}$$

or equivalently

$$\frac{n - \sigma_1}{1 + q} + \frac{n - \sigma_2}{1 + p} \leq n - \alpha. \tag{1.8}$$

We shall also assume that $p, q > 1$. Moreover, the theorem further asserts that there are also no positive integrable solutions in the supercritical case.

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Theorem 3. *Suppose $q \geq p > 1$, $\sigma_1 \geq \sigma_2$ (so that $q_0 \geq p_0$), and let u, v be positive solutions of (1.1) satisfying the non-subcritical condition (1.8).*
*(i) Then u, v are integrable solutions if and only if u, v are bounded and decay with the **fast rates** as $|x| \longrightarrow \infty$:*

$$
\begin{cases}
u(x) \simeq |x|^{-(n-\alpha)}; \\
v(x) \simeq |x|^{-(n-\alpha)}, & \text{if } p(n-\alpha) + \sigma_2 > n; \\
v(x) \simeq |x|^{-(n-\alpha)} \ln|x|, & \text{if } p(n-\alpha) + \sigma_2 = n; \\
v(x) \simeq |x|^{-(p(n-\alpha)-(\alpha-\sigma_2))}, & \text{if } p(n-\alpha) + \sigma_2 < n.
\end{cases}
$$

(ii) Suppose that, in addition, $\alpha \in (1, n)$. If u, v are integrable solutions of (1.1), then the critical condition,

$$
\frac{n - \sigma_1}{1 + q} + \frac{n - \sigma_2}{1 + p} = n - \alpha, \tag{1.9}
$$

necessarily holds.

Remark 4. *(i) Notice that if $\sigma_1 = \sigma_2 = 0$, the Hardy–Sobolev type system (1.1) coincides with the HLS system (1.4). Indeed, our main results on the integrable solutions, including the subsequent results below on the optimal integrability, boundedness, radial symmetry and monotonicity of solutions, coincide with and thus extend past results of [7, 20] (also see [4, 21, 22, 42] for closely related results).*

(ii) To further illustrate their connection, we remark that the integrable solutions in the unweighted case turn out to be the finite-energy solutions of the HLS system (1.4), i.e. the critical points $(u, v) \in L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)$ for the HLS functional.

(iii) In the weighted case, however, system (1.1) cannot be recovered from (1.4) due to their different singular weights. In fact, the asymptotic behavior of positive solutions between the two are <u>not</u> the same (cf. [23, 25]).

Our last main result asserts that if u, v are bounded and positive but are not integrable solutions, then they "decay with almost the slow rates" and we conjecture that they actually do converge with the slow rates. Of course, if this conjecture were true, then Theorem 2 would hold without any additional decaying assumption on solutions. However, if, in addition, u, v are decaying solutions, then they do indeed decay with the slow rates.

Theorem 4. *Let u, v be bounded positive solutions of (1.1). Then there hold the following.*

(i) *There does not exist a positive constant c such that*

$$\text{either } u(x) \geq c(1+|x|)^{-\theta_1} \text{ or } v(x) \geq c(1+|x|)^{-\theta_2},$$

where $\theta_1 < q_0$, $\theta_2 < p_0$.

(ii) *If u, v are not integrable solutions, then u, v decay with rates not faster than the slow rates. Namely, there does not exist a positive constant C such that*

$$\text{either } u(x) \leq C(1+|x|)^{-\theta_3} \text{ or } v(x) \leq C(1+|x|)^{-\theta_4},$$

where $\theta_3 > q_0$, $\theta_4 > p_0$.

(iii) *If u, v are not integrable solutions but are decaying solutions, then u, v must necessarily have the slow rates as $|x| \longrightarrow \infty$.*

Remark 5. *For the sake of conciseness, rather than formally stating the corresponding results for the poly-harmonic system (1.2) as corollaries, we only point out that our main results for the integral system do translate over to system (1.2) provided that the equivalence conditions described earlier are satisfied.*

The remaining parts of this paper are organized in the following way. In section 2, some preliminary results are established in which Theorem 1 is an immediate consequence of. Then, we apply an integral form of a Pohozaev type identity to prove Theorem 2. Section 3 establishes several key properties of integrable solutions: the optimal integrability, boundedness and convergence properties of integrable solutions, and we show the integrable solutions are radially symmetric and decreasing about the origin. Using these properties, we prove Theorem 3 in section 4. The paper then concludes with section 5, which contains the proof of Theorem 4.

2 Slow decay rates and the non-existence theorem

Proposition 1. *Let u, v be bounded and decaying positive solutions of (1.1).*

(i) *There exists a positive constant C such that as $|x| \longrightarrow \infty$*

$$u(x) \leq C|x|^{-q_0} \text{ and } v(x) \leq C|x|^{-p_0}.$$

(ii) Moreover, the improper integrals,

$$\int_{\mathbb{R}^n} \frac{u(x)^{p+1}}{|x|^{\sigma_2}} \, dx \ \ and \ \ \int_{\mathbb{R}^n} \frac{v(x)^{q+1}}{|x|^{\sigma_1}} \, dx,$$

are finite provided that the subcritical condition (1.6) holds.

Proof. For $|x| > 2R$ with $R > 0$ suitably large,

$$u(x) \geq \int_{B_{|x|}(0) \backslash B_R(0)} \frac{v(y)^q}{|x-y|^{n-\alpha}|y|^{\sigma_1}} \, dy$$

$$\geq Cv(x)^q|x|^{\alpha-\sigma_1-n} \int_{B_{|x|}(0)\backslash B_R(0)} dt \geq Cv(x)^q|x|^{\alpha-\sigma_1}. \tag{2.1}$$

Similarly, we can show

$$v(x) \geq Cu(x)^p|x|^{\alpha-\sigma_2}, \tag{2.2}$$

and combining (2.1) with (2.2) gives us

$$u(x) \geq Cv(x)^q|x|^{\alpha-\sigma_1} \geq Cu(x)^{pq}|x|^{q(\alpha-\sigma_2)+(\alpha-\sigma_1)},$$
$$v(x) \geq Cu(x)^p|x|^{\alpha-\sigma_2} \geq Cv(x)^{pq}|x|^{p(\alpha-\sigma_1)+(\alpha-\sigma_2)}.$$

Indeed, these estimates imply

$$u(x) \leq C|x|^{-q_0} \ \ and \ \ v(x) \leq C|x|^{-p_0} \ \ as \ \ |x| \longrightarrow \infty.$$

In addition,

$$\int_{\mathbb{R}^n} \frac{u(x)^{p+1}}{|x|^{\sigma_2}} \, dx \leq \int_{B_R(0)} \frac{u(x)^{p+1}}{|x|^{\sigma_2}} \, dx + \int_{B_R(0)^C} \frac{u(x)^{p+1}}{|x|^{\sigma_2}} \, dx$$

$$\leq C_1 + C_2 \int_R^\infty t^{-q_0(p+1)+n-\sigma_2} \frac{dt}{t} < \infty,$$

since (1.6) implies $-q_0(p+1) + n - \sigma_2 = n - \alpha - (q_0 + p_0) < 0$. Likewise, $\int_{\mathbb{R}^n} \frac{v(x)^{q+1}}{|x|^{\sigma_1}} \, dx < \infty$ using similar calculations, and this completes the proof. $\square$

Proposition 2. *Let u, v be bounded positive solutions of (1.1). Then there exists a positive constant $C > 0$ such that as $|x| \longrightarrow \infty$,*

$$u(x) \geq \frac{C}{(1+|x|)^{n-\alpha}} \ \ and \ \ v(x) \geq \frac{C}{(1+|x|)^{\min\{n-\alpha, \, p(n-\alpha)-(\alpha-\sigma_2)\}}}.$$

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Proof. For $y \in B_1(0)$, we can find a $C > 0$ such that

$$C \leq \int_{B_1(0)} \frac{v(y)^q}{|y|^{\sigma_1}} \, dy, \int_{B_1(0)} \frac{u(y)^p}{|y|^{\sigma_2}} \, dy < \infty.$$

Thus for $x \in B_1(0)^C$, we have

$$u(x) \geq \int_{B_1(0)} \frac{v(y)^q}{|x-y|^{n-\alpha}|y|^{\sigma_1}} \, dy$$

$$\geq \frac{C}{(1+|x|)^{n-\alpha}} \int_{B_1(0)} \frac{v(y)^q}{|y|^{\sigma_1}} \, dy \geq \frac{C}{(1+|x|)^{n-\alpha}}. \qquad (2.3)$$

Similarly, we can show

$$v(x) \geq \frac{C}{(1+|x|)^{n-\alpha}}.$$

Then, with the help of estimate (2.3), we get

$$v(x) \geq \int_{B_{|x|/2}(x)} \frac{u(y)^p}{|x-y|^{n-\alpha}|y|^{\sigma_2}} \, dy$$

$$\geq \frac{C}{(1+|x|)^{p(n-\alpha)+\sigma_2}} \int_0^{|x|/2} t^\alpha \frac{dt}{t} = \frac{C}{(1+|x|)^{p(n-\alpha)-(\alpha-\sigma_2)}}.$$

$\square$

Proof of Theorem 1. This is a direct consequence of Proposition 1(i) and Proposition 2. $\square$

Proof of Theorem 2. We proceed by contradiction. That is, assume u, v are bounded and decaying positive classical solutions. First, notice that integration by parts implies

$$\int_{B_R(0)} \frac{v(x)^q}{|x|^{\sigma_1}}(x \cdot \nabla v(x)) + \frac{u(x)^p}{|x|^{\sigma_2}}(x \cdot \nabla u(x)) \, dx$$

$$= \frac{1}{1+q} \int_{B_R(0)} \frac{x}{|x|^{\sigma_1}} \cdot \nabla(v(x)^{q+1}) \, dx + \frac{1}{1+p} \int_{B_R(0)} \frac{x}{|x|^{\sigma_2}} \cdot \nabla(u(x)^{p+1}) \, dx$$

$$= -\frac{n-\sigma_1}{1+q} \int_{B_R(0)} \frac{v(x)^{q+1}}{|x|^{\sigma_1}} \, dx - \frac{n-\sigma_2}{1+p} \int_{B_R(0)} \frac{u(x)^{p+1}}{|x|^{\sigma_2}} \, dx$$

$$+\frac{R}{1+q} \int_{\partial B_R(0)} \frac{v(x)^{q+1}}{|x|^{\sigma_1}} \, ds + \frac{R}{1+p} \int_{\partial B_R(0)} \frac{u(x)^{p+1}}{|x|^{\sigma_2}} \, ds.$$

Note that this identity follows more precisely by integrating on $B_R(0) \backslash B_\varepsilon(0)$ then sending $\varepsilon \longrightarrow 0$ after the appropriate calculations. Then, by virtue of Proposition 1(ii), we can find a sequence $\{R_j\}$ such that as $R_j \longrightarrow \infty$,

$$R_j \int_{\partial B_{R_j}(0)} \frac{v(x)^{q+1}}{|x|^{\sigma_1}} \, ds, \; R_j \int_{\partial B_{R_j}(0)} \frac{u(x)^{p+1}}{|x|^{\sigma_2}} \, ds \longrightarrow 0.$$

Thus, we obtain the identity

$$\int_{\mathbb{R}^n} \frac{v(x)^q}{|x|^{\sigma_1}} (x \cdot \nabla v(x)) + \frac{u(x)^p}{|x|^{\sigma_2}} (x \cdot \nabla u(x)) \, dx$$

$$= -\left\{ \frac{n - \sigma_1}{1 + q} + \frac{n - \sigma_2}{1 + p} \right\} \int_{\mathbb{R}^n} \frac{v(x)^{q+1}}{|x|^{\sigma_1}} \, dx, \quad (2.4)$$

where we used the fact that

$$\int_{\mathbb{R}^n} \frac{u(x)^{p+1}}{|x|^{\sigma_2}} \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(x)^p v(z)^q}{|x - z|^{n-\alpha} |x|^{\sigma_2} |z|^{\sigma_1}} \, dz dx = \int_{\mathbb{R}^n} \frac{v(x)^{q+1}}{|x|^{\sigma_1}} \, dx.$$

From the first equation with $\lambda \neq 0$, we write

$$u(\lambda x) = \int_{\mathbb{R}^n} \frac{v(y)^q}{|\lambda x - y|^{n-\alpha} |y|^{\sigma_1}} \, dy = \lambda^{\alpha - \sigma_1} \int_{\mathbb{R}^n} \frac{v(\lambda z)^q}{|x - z|^{n-\alpha} |z|^{\sigma_1}} \, dz.$$

Differentiating this rescaled equation with respect to λ then taking $\lambda = 1$ gives us

$$x \cdot \nabla u(x) = (\alpha - \sigma_1) \int_{\mathbb{R}^n} \frac{v(z)^q}{|x - z|^{n-\alpha} |z|^{\sigma_1}} \, dz + \int_{\mathbb{R}^n} \frac{qv(z)^{q-1}(z \cdot \nabla v)}{|x - z|^{n-\alpha} |z|^{\sigma_1}} \, dz$$

$$= (\alpha - \sigma_1) u(x) + \int_{\mathbb{R}^n} \frac{z \cdot \nabla v(z)^q}{|x - z|^{n-\alpha} |z|^{\sigma_1}} \, dz \; (x \neq 0). \quad (2.5)$$

Note that an integration by parts yields

$$\int_{B_R(0)} \frac{z \cdot \nabla v(z)^q}{|x - z|^{n-\alpha} |z|^{\sigma_1}} \, dz$$

$$= R \int_{\partial B_R(0)} \frac{v(z)^q}{|x - z|^{n-\alpha} |z|^{\sigma_1}} \, ds - (n - \sigma_1) \int_{B_R(0)} \frac{v(z)^q}{|x - z|^{n-\alpha} |z|^{\sigma_1}} \, dz$$

$$- (n - \alpha) \int_{B_R(0)} \frac{(z \cdot (x - z)) v(z)^q}{|x - z|^{n-\alpha+2} |z|^{\sigma_1}} \, dz.$$

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By virtue of $\int_{\mathbb{R}^n} \frac{v(y)^q}{|x-z|^{n-\alpha}|z|^{\sigma_1}} dz < \infty$, we can find a sequence $\{R_j\}$ such that

$$R_j \int_{\partial B_{R_j}(0)} \frac{v(z)^q}{|x-z|^{n-\alpha}|z|^{\sigma_1}} ds \longrightarrow 0 \ \text{ as } \ R_j \longrightarrow \infty$$

and thus obtain

$$\int_{\mathbb{R}^n} \frac{z \cdot \nabla v(z)^q}{|x-z|^{n-\alpha}|z|^{\sigma_1}} dz = -(n-\sigma_1) \int_{\mathbb{R}^n} \frac{v(z)^q}{|x-z|^{n-\alpha}|z|^{\sigma_1}} dz$$
$$- (n-\alpha) \int_{\mathbb{R}^n} \frac{(z \cdot (x-z))v(z)^q}{|x-z|^{n-\alpha+2}|z|^{\sigma_1}} dz.$$

Hence, inserting this into (2.5) yields

$$x \cdot \nabla u(x) = -(n-\alpha)u(x) - (n-\alpha) \int_{\mathbb{R}^n} \frac{(z \cdot (x-z))v(z)^q}{|x-z|^{n-\alpha+2}|z|^{\sigma_1}} dz. \qquad (2.6)$$

Similar calculations on the second integral equation will lead to

$$x \cdot \nabla v(x) = -(n-\alpha)v(x) - (n-\alpha) \int_{\mathbb{R}^n} \frac{(z \cdot (x-z))u(z)^p}{|x-z|^{n-\alpha+2}|z|^{\sigma_2}} dz. \qquad (2.7)$$

Now multiply (2.6) and (2.7) by $|x|^{-\sigma_2}u(x)^p$ and $|x|^{-\sigma_1}v(x)^q$, respectively, sum the resulting equations together and integrate over $\mathbb{R}^n$ to get

$$\int_{\mathbb{R}^n} \frac{v(x)^q}{|x|^{\sigma_1}} (x \cdot \nabla v(x))\, dx + \int_{\mathbb{R}^n} \frac{u(x)^p}{|x|^{\sigma_2}} (x \cdot \nabla u(x))\, dx$$
$$= -(n-\alpha)\left\{ \int_{\mathbb{R}^n} \frac{v(x)^{q+1}}{|x|^{\sigma_1}} + \frac{u(x)^{p+1}}{|x|^{\sigma_2}}\, dx \right\}$$
$$- (n-\alpha) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(z \cdot (x-z) + x \cdot (z-x))u(z)^p v(x)^q}{|x-z|^{n-\alpha+2}|z|^{\sigma_2}|x|^{\sigma_1}}\, dzdx.$$

By noticing that $z \cdot (x-z) + x \cdot (z-x) = -|x-z|^2$, we obtain the Pohozaev type identity

$$\int_{\mathbb{R}^n} \frac{v(x)^q}{|x|^{\sigma_1}} (x \cdot \nabla v(x)) + \frac{u(x)^p}{|x|^{\sigma_2}} (x \cdot \nabla u(x))\, dx = -(n-\alpha) \int_{\mathbb{R}^n} \frac{v(x)^{q+1}}{|x|^{\sigma_1}}\, dx. \quad (2.8)$$

Inserting (2.4) into (2.8) yields

$$\left\{ \frac{n-\sigma_1}{1+q} + \frac{n-\sigma_2}{1+p} - (n-\alpha) \right\} \int_{\mathbb{R}^n} \frac{v(x)^{q+1}}{|x|^{\sigma_1}}\, dx = 0,$$

but this contradicts with (1.6). This completes the proof of the theorem. $\qquad \square$

3 Properties of integrable solutions

3.1 An equivalent form of the weighted HLS inequality

The following estimate is a consequence of the doubly weighted HLS inequality by duality, and it is the version of the weighted HLS inequality we apply in this paper.

Lemma 1. *Let $p, q \in (1, \infty)$, $\alpha \in (0, n)$ and $0 \le \sigma_1 + \sigma_2 \le \alpha$, and define*

$$I_\alpha f(x) \doteq \int_{\mathbb{R}^n} \frac{f(y)}{|x|^{\sigma_1} |x - y|^{n-\alpha} |y|^{\sigma_2}} \, dy.$$

Then

$$\|I_\alpha f\|_{L^q(\mathbb{R}^n)} \le C_{\sigma_i, p, \alpha, n} \|f\|_{L^p(\mathbb{R}^n)},$$

where $\frac{1}{p} - \frac{1}{q} = \frac{\alpha - (\sigma_1 + \sigma_2)}{n}$ and $\frac{1}{q} - \frac{n-\alpha}{n} < \frac{\sigma_1}{n} < \frac{1}{q}$.

3.2 Integrability of solutions

Theorem 5. *Suppose $q \ge p > 1$ and $\sigma_1 \ge \sigma_2$. If u and v are positive integrable solutions of (1.1), then $(u, v) \in L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n)$ for each pair (r, s) such that*

$$\frac{1}{r} \in \left(0, \frac{n - \alpha}{n} \right) \quad and \quad \frac{1}{s} \in \left(0, \min \left\{ \frac{n - \alpha}{n}, \frac{p(n - \alpha) - (\alpha - \sigma_2)}{n} \right\} \right).$$

Proof. Step 1: Establish an initial interval of integrability.

Set $a = \frac{1}{r_0} = \frac{\alpha(1+q) - (\sigma_1 + \sigma_2 q)}{n(pq-1)}$, $b = \frac{1}{s_0} = \frac{\alpha(1+p) - (\sigma_2 + \sigma_1 p)}{n(pq-1)}$ and let $\frac{1}{r} \in (a - b, \frac{n-\alpha}{n})$ and $\frac{1}{s} \in (0, \frac{n-\alpha}{n} - a + b)$ such that

$$\frac{1}{r} - \frac{1}{s} = \frac{1}{r_0} - \frac{1}{s_0}.$$

Thus, we have

$$\frac{1}{r} + \frac{\alpha - \sigma_1}{n} = \frac{q - 1}{s_0} + \frac{1}{s} \quad \text{and} \quad \frac{1}{s} + \frac{\alpha - \sigma_2}{n} = \frac{p - 1}{r_0} + \frac{1}{r}. \tag{3.1}$$

Let $A > 0$ and define $u_A = u$ if $u > A$ or $|x| > A$, $u_A = 0$ if $u \le A$ and $|x| \le A$; we give v_A the analogous definition. Consider the integral operator $T = (T_1, T_2)$ where

$$T_1 g(x) = \int_{\mathbb{R}^n} \frac{v_A(y)^{q-1} g(y)}{|x - y|^{n-\alpha} |y|^{\sigma_1}} \, dy \quad \text{and} \quad T_2 f(x) = \int_{\mathbb{R}^n} \frac{u_A(y)^{p-1} f(y)}{|x - y|^{n-\alpha} |y|^{\sigma_2}} \, dy$$

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for $f \in L^r(\mathbb{R}^n)$ and $g \in L^s(\mathbb{R}^n)$. By virtue of (3.1), applying the weighted Hardy–Littlewood–Sobolev inequality followed by Hölder's inequality gives us

$$\|T_1 g\|_{L^r} \le C\|v_A^{q-1} g\|_{\frac{nr}{n+r(\alpha-\sigma_1)}} \le C\|v_A\|_{s_0}^{q-1}\|g\|_s,$$
$$\|T_2 f\|_{L^s} \le C\|u_A^{p-1} f\|_{\frac{ns}{n+s(\alpha-\sigma_2)}} \le C\|u_A\|_{r_0}^{p-1}\|f\|_r.$$

We may choose A sufficiently large so that

$$C\|v_A\|_{s_0}^{q-1}, \ C\|u_A\|_{r_0}^{p-1} \le 1/2$$

and the operator $T(f,g) = (T_1 g, T_2 f)$, equipped with the norm

$$\|(f_1, f_2)\|_{L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n)} = \|f_1\|_r + \|f_2\|_s,$$

is a contraction map from $L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n)$ to itself

$$\text{for all } (\frac{1}{r}, \frac{1}{s}) \in I \doteq (a-b, \frac{n-\alpha}{n}) \times (0, \frac{n-\alpha}{n} - a + b).$$

Thus, T is also a contraction map from $L^{r_0}(\mathbb{R}^n) \times L^{s_0}(\mathbb{R}^n)$ to itself since $(\frac{1}{r_0}, \frac{1}{s_0}) \in I$. Now define

$$F = \int_{\mathbb{R}^n} \frac{(v - v_A)^q(y)}{|x-y|^{n-\alpha}|y|^{\sigma_1}} \, dy \ \text{ and } \ G = \int_{\mathbb{R}^n} \frac{(u - u_A)^p(y)}{|x-y|^{n-\alpha}|y|^{\sigma_2}} \, dy.$$

Then the weighted HLS inequality implies that $(F, G) \in L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n)$ and since (u, v) satisfies

$$(f, g) = T(f, g) + (F, G),$$

applying Lemma 2.1 from [18] yields

$$(u, v) \in L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n) \text{ for all } (\frac{1}{r}, \frac{1}{s}) \in I. \tag{3.2}$$

Step 2: Extend the interval of integrability I.

First, we claim that

$$q\left\{\frac{n-\alpha}{n} - a + b\right\} - \frac{\alpha - \sigma_1}{n} > a - b. \tag{3.3}$$

If this holds true, we can then apply the weighted HLS inequality to obtain

$$\|u\|_r \le C\|v^q\|_{\frac{nr}{n+r(\alpha-\sigma_1)}} \le C\|v\|_{\frac{nrq}{n+r(\alpha-\sigma_1)}}^q.$$

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Then, since $v \in L^s(\mathbb{R}^n)$ for all $\frac{1}{s} \in (0, \frac{n-\alpha}{n} - a + b)$, we obtain that $u \in L^r(\mathbb{R}^n)$ for $\frac{1}{r} \in (0, q\{\frac{n-\alpha}{n} - a + b\} - \frac{\alpha - \sigma_1}{n})$, where we are using (3.3) for the last interval to make sense. Combining this with (3.2) yields

$$u \in L^r(\mathbb{R}^n) \text{ for all } \frac{1}{r} \in \left(0, \frac{n-\alpha}{n}\right). \tag{3.4}$$

Likewise, the weighted HLS inequality implies

$$\|v\|_s \le C\|u\|^p_{\frac{nsp}{n+s(\alpha-\sigma_2)}}.$$

Thus, combining this with (3.4) yields

$$v \in L^s(\mathbb{R}^n) \text{ for all } \frac{1}{s} \in \left(0, \min\left\{\frac{n-\alpha}{n}, \frac{p(n-\alpha)-(\alpha-\sigma_2)}{n}\right\}\right).$$

It remains to verify the claim (3.3). To do so, notice that (1.7) implies that

$$q\left\{\frac{n-\alpha}{n} - a + b\right\} - \frac{\alpha-\sigma_1}{n} > qb - \frac{\alpha-\sigma_1}{n} = \frac{\alpha q - \sigma_2 q + \alpha - \sigma_1}{n(pq-1)}$$

$$= \frac{\alpha(q-p) + \alpha p - \sigma_2 q + \alpha - \sigma_1}{n(pq-1)} > \frac{\alpha(q-p) + \sigma_1 p - \sigma_2 q + \sigma_2 - \sigma_1}{n(pq-1)}$$

$$= \frac{\alpha(q-p) - \sigma_1(1-p) - \sigma_2(q-1)}{n(pq-1)} = a - b.$$

This completes the proof. $\qquad\square$

Remark 6. *Actually, it is not too difficult to show that the interval of integrability of Theorem 5 is optimal. That is, $\|u\|_r = \infty$ and $\|v\|_s = \infty$ at the endpoints*

$$r = \frac{n}{n-\alpha} \text{ and } s = \max\left\{\frac{n}{n-\alpha}, \frac{n}{p(n-\alpha)-(\alpha-\sigma_2)}\right\}.$$

3.3 Integrable solutions are ground states

Theorem 6. *If u, v are positive integrable solutions of (1.1), then u, v are bounded and converge to zero as $|x| \longrightarrow \infty$.*

Proof. We prove this in two steps. The first step shows the boundedness of integrable solutions and the second step verifies the decay property.
Part 1: u and v are in $L^\infty(\mathbb{R}^n)$.

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By exchanging the order of integration and choosing a suitably small $c > 0$, we can write

$$u(x) \le C\Big(\int_0^c \frac{\int_{B_t(x)} |y|^{-\sigma_1} v(y)^q \, dy}{t^{n-\alpha}} \, \frac{dt}{t} + \int_c^\infty \frac{\int_{B_t(x)} |y|^{-\sigma_1} v(y)^q \, dy}{t^{n-\alpha}} \, \frac{dt}{t} \Big)$$
$$\doteq I_1 + I_2.$$

(i) We estimate I_1 first and assume $|x| > 1$, since the case where $|x| \le 1$ can be treated similarly. Then Hölder's inequality yields

$$\int_{B_t(x)} \frac{v(y)^q}{|y|^{\sigma_1}} \, dy \le Ct^{-\sigma_1} |B_t(x)|^{1-1/\ell} \|v^q\|_\ell$$

for $\ell > 1$. Then choose ℓ suitably large so that $\varepsilon q = 1/\ell$ is sufficiently small and so $v^q \in L^\ell(\mathbb{R}^n)$ as a result of Theorem 5. Thus,

$$I_1 \le C \|v^q\|_\ell \int_0^c \frac{|B_t(x)|^{1-\varepsilon q}}{t^{n-\alpha+\sigma_1}} \, \frac{dt}{t} \le C \int_0^c t^{\alpha-\sigma_1-nq\varepsilon} \, \frac{dt}{t} < \infty.$$

(ii) If $z \in B_\delta(x)$, then $B_t(x) \subset B_{t+\delta}(z)$. From this, observe that for $\delta \in (0,1)$ and $z \in B_\delta(x)$,

$$I_2 \le C \int_c^\infty \frac{\int_{B_t(x)} |y|^{-\sigma_1} v(y)^q \, dy}{t^{n-\alpha}} \, \frac{dt}{t}$$
$$\le C \int_c^\infty \frac{\int_{B_{t+\delta}(z)} |y|^{-\sigma_1} v(y)^q \, dy}{(t+\delta)^{n-\alpha}} \Big(\frac{t+\delta}{t} \Big)^{n-\alpha+1} \frac{d(t+\delta)}{t+\delta}$$
$$\le C(1+\delta)^{n-\alpha+1} \int_{c+\delta}^\infty \frac{\int_{B_t(z)} |y|^{-\sigma_1} v(y)^q \, dy}{t^{n-\alpha}} \, \frac{dt}{t} \le Cu(z).$$

These estimates for I_1 and I_2 yield

$$u(x) \le C_1 + C_2 u(z) \text{ for } z \in B_\delta(x).$$

Integrating this estimate on $B_\delta(x)$ then applying Hölder's inequality gives us

$$u(x) \le C_1 + C_2 |B_\delta(x)|^{-1} \int_{B_\delta(x)} u(z) \, dz \le C_1 + C_2 |B_\delta(x)|^{-1/r_0} \|u\|_{r_0} < \infty.$$

Hence, u is bounded in $\mathbb{R}^n$. Using similar calculations, we can also show v is bounded in $\mathbb{R}^n$.

Part 2: $u(x), v(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

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Choose $x \in \mathbb{R}^n$. Then for each $\varepsilon > 0$, there exists a sufficiently small $\delta > 0$ such that

$$\int_0^\delta \frac{\int_{B_t(x)} |y|^{-\sigma_1} v(y)^q \, dy}{t^{n-\alpha}} \frac{dt}{t} \leq C\|v\|_\infty^q \int_0^\delta t^{\alpha-\sigma_1} \frac{dt}{t} < \varepsilon.$$

Likewise, for $|x - z| < \delta$, we have

$$\int_\delta^\infty \frac{\int_{B_t(x)} |y|^{-\sigma_1} v(y)^q \, dy}{t^{n-\alpha}} \frac{dt}{t}$$

$$\leq \int_\delta^\infty \frac{\int_{B_{t+\delta}(z)} |y|^{-\sigma_1} v(y)^q \, dy}{(t+\delta)^{n-\alpha}} \left(\frac{t+\delta}{t}\right)^{n-\alpha+1} \frac{d(t+\delta)}{t+\delta}$$

$$\leq C \int_0^\infty \frac{\int_{B_t(z)} |y|^{-\sigma_1} v(y)^q \, dy}{t^{n-\alpha}} \frac{dt}{t} = Cu(z).$$

Therefore, the last two estimates imply

$$u(x) \leq \varepsilon + Cu(z) \quad \text{for} \quad z \in B_\delta(x),$$

and since $u \in L^{r_0}(\mathbb{R}^n)$, we obtain

$$u(x)^{r_0} = \frac{1}{|B_\delta(x)|} \int_{B_\delta(x)} u(x)^{r_0} \, dz \leq C_1 \varepsilon^{r_0} + C_2 \frac{1}{|B_\delta(x)|} \int_{B_\delta(x)} u(z)^{r_0} \, dz \longrightarrow 0$$

as $|x| \longrightarrow \infty$ and $\varepsilon \longrightarrow 0$. Hence, $\lim_{|x|\longrightarrow\infty} u(x) = 0$, and similar calculations will show $\lim_{|x|\longrightarrow\infty} v(x) = 0$. This completes the proof. $\square$

3.4 Radial symmetry and monotonicity

Theorem 7. *Let u, v be positive integrable solutions of (1.1). Then u and v are radially symmetric and monotone decreasing about the origin.*

We employ the integral form of the method of moving planes (cf. [9, 11]) to prove this result, but first, we introduce some preliminary tools. For $\lambda \in \mathbb{R}$, define

$$\Sigma_\lambda \doteq \{x = (x_1, \ldots, x_n) \,|\, x_1 \geq \lambda\},$$

let $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$ be the reflection of x across the plane $\Gamma_\lambda \doteq \{x_1 = \lambda\}$ and let $u_\lambda(x) = u(x^\lambda)$ and $v_\lambda(x) = v(x^\lambda)$.

Lemma 2. *There holds*

$$u_\lambda(x) - u(x) = \int_{\Sigma_\lambda} \Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \Big) \frac{1}{|y^\lambda|^{\sigma_1}} (v_\lambda(y)^q - v(y)^q) \, dy$$

$$- \int_{\Sigma_\lambda} \Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \Big) \Big(\frac{1}{|y|^{\sigma_1}} - \frac{1}{|y^\lambda|^{\sigma_1}} \Big) v(y)^q \, dy;$$

$$v_\lambda(x) - v(x) = \int_{\Sigma_\lambda} \Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \Big) \frac{1}{|y^\lambda|^{\sigma_2}} (u_\lambda(y)^p - u(y)^p) \, dy$$

$$- \int_{\Sigma_\lambda} \Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \Big) \Big(\frac{1}{|y|^{\sigma_2}} - \frac{1}{|y^\lambda|^{\sigma_2}} \Big) u(y)^p \, dy.$$

Proof. We only prove the first identity since the second identity follows similarly. By noticing $|x - y^\lambda| = |x^\lambda - y|$, we obtain

$$u_\lambda(x) = \int_{\Sigma_\lambda} \frac{v(y)^q}{|x^\lambda - y|^{n-\alpha}|y|^{\sigma_1}} \, dy + \int_{\Sigma_\lambda} \frac{v_\lambda(y)^q}{|x - y|^{n-\alpha}|y^\lambda|^{\sigma_1}} \, dy,$$

$$u(x) = \int_{\Sigma_\lambda} \frac{v(y)^q}{|x - y|^{n-\alpha}|y|^{\sigma_1}} \, dy + \int_{\Sigma_\lambda} \frac{v_\lambda(y)^q}{|x^\lambda - y|^{n-\alpha}|y^\lambda|^{\sigma_1}} \, dy.$$

Taking their difference yields

$$u_\lambda(x) - u(x) = \int_{\Sigma_\lambda} \Big(\frac{v_\lambda(y)^q}{|x-y|^{n-\alpha}|y^\lambda|^{\sigma_1}} - \frac{v_\lambda(y)^q}{|x^\lambda - y|^{n-\alpha}|y^\lambda|^{\sigma_1}} \Big) \, dy$$

$$- \int_{\Sigma_\lambda} \Big(\frac{v(y)^q}{|x-y|^{n-\alpha}|y|^{\sigma_1}} - \frac{v(y)^q}{|x^\lambda - y|^{n-\alpha}|y|^{\sigma_1}} \Big) \, dy$$

$$= \int_{\Sigma_\lambda} \Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \Big) \frac{v_\lambda(y)^q}{|y^\lambda|^{\sigma_1}} \, dy$$

$$- \int_{\Sigma_\lambda} \Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \Big) \frac{v(y)^q}{|y|^{\sigma_1}} \, dy,$$

and the result follows accordingly. □

Proof of Theorem 7. **Step 1:** We claim that there exists $N > 0$ such that if $\lambda \le -N$, there hold

$$u_\lambda(x) \le u(x) \quad \text{and} \quad v_\lambda(x) \le v(x). \tag{3.5}$$

Define $\Sigma_\lambda^u \doteq \{x \in \Sigma_\lambda \,|\, u_\lambda(x) > u(x)\}$ and $\Sigma_\lambda^v \doteq \{x \in \Sigma_\lambda \,|\, v_\lambda(x) > v(x)\}$. From Lemma 2, the mean-value theorem and since

$$\Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \Big) \Big(\frac{1}{|y|^{\sigma_1}} - \frac{1}{|y^\lambda|^{\sigma_1}} \Big) \ge 0 \quad \text{for } y \in \Sigma_\lambda,$$

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$$u_\lambda(x) - u(x) \le \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) \frac{1}{|y^\lambda|^{\sigma_1}} (v_\lambda(y)^q - v(y)^q)\, dy$$

$$\le \int_{\Sigma_\lambda^v} \frac{q v_\lambda(y)^{q-1}}{|x-y|^{n-\alpha} |y^\lambda|^{\sigma_1}} (v_\lambda(y) - v(y))\, dy.$$

Thus, applying the weighted HLS inequality followed by Hölder's inequality gives us

$$\|u_\lambda - u\|_{L^{r_0}(\Sigma_\lambda^u)} \le C \|v_\lambda^{q-1}(v_\lambda - v)\|_{L^{\frac{nr_0}{n+r_0(\alpha-\sigma_1)}}(\Sigma_\lambda^v)}$$

$$\le C \|v_\lambda\|_{L^{s_0}(\Sigma_\lambda^v)}^{q-1} \|v_\lambda - v\|_{L^{s_0}(\Sigma_\lambda^v)} \le C \|v\|_{L^{s_0}(\Sigma_\lambda^C)}^{q-1} \|v_\lambda - v\|_{L^{s_0}(\Sigma_\lambda^v)}. \quad (3.6)$$

Similarly, there holds

$$\|v_\lambda - v\|_{L^{s_0}(\Sigma_\lambda^v)} \le C \|u_\lambda^{p-1}(u_\lambda - u)\|_{L^{\frac{ns_0}{n+s_0(\alpha-\sigma_2)}}(\Sigma_\lambda^u)}$$

$$\le C \|u_\lambda\|_{L^{r_0}(\Sigma_\lambda^u)}^{p-1} \|u_\lambda - u\|_{L^{r_0}(\Sigma_\lambda^u)} \le C \|u\|_{L^{r_0}(\Sigma_\lambda^C)}^{p-1} \|u_\lambda - u\|_{L^{r_0}(\Sigma_\lambda^u)}. \quad (3.7)$$

By virtue of $(u,v) \in L^{r_0}(\mathbb{R}^n) \times L^{s_0}(\mathbb{R}^n)$, we can choose N suitably large such that for $\lambda \le -N$, there holds

$$C^2 \|u\|_{L^{r_0}(\Sigma_\lambda^C)}^{p-1} \|v\|_{L^{s_0}(\Sigma_\lambda^C)}^{q-1} \le 1/2.$$

Therefore, combining (3.6) with (3.7) gives us

$$\begin{cases} \|u_\lambda - u\|_{L^{r_0}(\Sigma_\lambda^u)} \le \frac{1}{2} \|u_\lambda - u\|_{L^{r_0}(\Sigma_\lambda^u)}, \\ \|v_\lambda - v\|_{L^{s_0}(\Sigma_\lambda^v)} \le \frac{1}{2} \|v_\lambda - v\|_{L^{s_0}(\Sigma_\lambda^v)}, \end{cases}$$

which further implies that $\|u_\lambda - u\|_{L^{r_0}(\Sigma_\lambda^u)} = 0$ and $\|v_\lambda - v\|_{L^{s_0}(\Sigma_\lambda^v)} = 0$. Thus, both Σ_λ^u and Σ_λ^v have measure zero and are therefore empty. This concludes the proof of the first claim.

Step 2: We can move the plane Γ_λ to the right provided that (3.5) holds. Let

$$\lambda_0 \doteq \sup\{\lambda \,|\, u_\lambda(x) \le u(x), v_\lambda(x) \le v(x)\}.$$

We claim that u, v are symmetric about the plane Γ_{λ_0}, i.e.

$$u_{\lambda_0}(x) = u(x) \text{ and } v_{\lambda_0}(x) = v(x) \text{ for all } x \in \Sigma_{\lambda_0}.$$

On the contrary, assume $\lambda_0 \le 0$ and for all $x \in \Sigma_{\lambda_0}$,

$$u_{\lambda_0}(x) \le u(x) \text{ and } v_{\lambda_0}(x) \le v(x), \text{ but } u_{\lambda_0}(x) \not\equiv u(x) \text{ or } v_{\lambda_0}(x) \not\equiv v(x).$$

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But we will show this implies the plane can be moved further to the right, thereby contradicting the definition of λ_0. Namely, there is a small $\varepsilon > 0$ such that

$$u_\lambda(x) \leq u(x) \ \text{ and } \ v_\lambda(x) \leq v(x), \quad x \in \Sigma_\lambda, \ \text{ for all } \ \lambda \in [\lambda_0, \lambda_0 + \varepsilon). \quad (3.8)$$

In the case, say $v_{\lambda_0}(x) \not\equiv v(x)$ on Σ_{λ_0}, Lemma 2 indicates $u_{\lambda_0}(x) < u(x)$ in the interior of Σ_{λ_0}. Define

$$\Phi^u_{\lambda_0} \doteq \{x \in \Sigma_{\lambda_0} \,|\, u_{\lambda_0}(x) \geq u(x)\} \ \text{ and } \ \Phi^v_{\lambda_0} \doteq \{x \in \Sigma_{\lambda_0} \,|\, v_{\lambda_0}(x) \geq v(x)\}.$$

Then $\Phi^u_{\lambda_0}$ and $\Phi^v_{\lambda_0}$ have measure zero and

$$\lim_{\lambda \to \lambda_0} \Sigma^u_\lambda \subset \Phi^u_{\lambda_0} \ \text{ and } \ \lim_{\lambda \to \lambda_0} \Sigma^v_\lambda \subset \Phi^v_{\lambda_0}.$$

Let Ω^* denote the reflection of the set Ω about the plane Γ_λ. According to the integrability of u and v, we can choose a suitably small ε such that for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$, $\|u\|_{L^{r_0}((\Sigma^u_\lambda)^*)}$ and $\|v\|_{L^{s_0}((\Sigma^v_\lambda)^*)}$ are sufficiently small. Therefore, (3.6) and (3.7) imply

$$\begin{cases} \|u_\lambda - u\|_{L^{r_0}(\Sigma^u_\lambda)} \leq C\|v\|^{q-1}_{L^{s_0}((\Sigma^v_\lambda)^*)}\|v_\lambda - v\|_{L^{s_0}(\Sigma^v_\lambda)} \leq \frac{1}{2}\|v_\lambda - v\|_{L^{s_0}(\Sigma^v_\lambda)}, \\[2mm] \|v_\lambda - v\|_{L^{s_0}(\Sigma^v_\lambda)} \leq C\|u\|^{p-1}_{L^{r_0}((\Sigma^u_\lambda)^*)}\|u_\lambda - u\|_{L^{r_0}(\Sigma^u_\lambda)} \leq \frac{1}{2}\|u_\lambda - u\|_{L^{r_0}(\Sigma^u_\lambda)}, \end{cases}$$

and we deduce that Σ^u_λ and Σ^v_λ are both empty. This proves (3.8). Hence, we conclude that u and v are symmetric and decreasing about the plane Γ_{λ_0}.

Step 3: We assert that u and v are radially symmetric and decreasing about the origin.

First, notice that λ_0 is indeed equal to zero. Otherwise, if $\lambda_0 < 0$, then Lemma 2 yields

$$\begin{aligned} 0 &= u(x) - u_{\lambda_0}(x) \\ &= \int_{\Sigma_{\lambda_0}} \Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^{\lambda_0}-y|^{n-\alpha}}\Big)\Big(\frac{1}{|y|^{\sigma_1}} - \frac{1}{|y^{\lambda_0}|^{\sigma_1}}\Big)v(y)^q \, dy \not\equiv 0, \end{aligned}$$

which is impossible. Therefore, u and v are symmetric and decreasing about the plane $\Gamma_{\lambda_0=0}$, and since the coordinate direction x_1 can be chosen arbitrarily, we conclude that u and v must be radially symmetric and decreasing about the origin. This completes the proof of the theorem. $\square$

4 Fast decay rates of integrable solutions

In this section, u, v are taken to be positive integrable solutions of system (1.1) unless further specified.

4.1 Fast decay rate for $u(x)$

Proposition 3. *There holds the following.*

(i) *The improper integral,* $A_0 \doteq \displaystyle\int_{\mathbb{R}^n} \frac{v(y)^q}{|y|^{\sigma_1}} \, dy < \infty$, *is convergent;*

(ii) $\displaystyle\lim_{|x| \longrightarrow \infty} u(x)|x|^{n-\alpha} = A_0$.

Remark 7. *According to this, we can find a large $R > 0$ such that*

$$u(x) = \frac{A_0 + o(1)}{|x|^{n-\alpha}} \quad for \ x \in \mathbb{R}^n \backslash B_R(0), \tag{4.1}$$

and we shall often invoke this property in establishing the decay rates for $v(x)$.

Proof. (i) Without loss of generality, we assume $\sigma_1 > 0$ since the proof for the unweighted case is similar but far simpler. For each $R > 0$, since $v \in L^\infty(\mathbb{R}^n)$ and $\sigma_1 < n$, we have

$$\int_{B_R(0)} \frac{v(y)^q}{|y|^{\sigma_1}} \, dy < \infty.$$

So it remains to show $\int_{B_R(0)^C} \frac{v(y)^q}{|y|^{\sigma_1}} \, dy < \infty$. There are two cases to consider.
(1.) Assume $n - \alpha \le p(n-\alpha) - (\alpha - \sigma_2)$. It is clear that $q \ge p$, $\sigma_1 \ge \sigma_2$ and (1.8) imply $q \ge \frac{n+\alpha-2\sigma_1}{n-\alpha}$. Choose $\varepsilon > 0$ with $\varepsilon \in (\alpha - 2\sigma_1, \alpha - \sigma_1)$. Then set $\ell = \frac{n+\varepsilon}{n+\alpha-2\sigma_1}$ and $\ell' = \frac{n+\varepsilon}{\varepsilon-\alpha+2\sigma_1}$ so that $\frac{1}{\ell} + \frac{1}{\ell'} = 1$, $lq > \frac{n}{n-\alpha}$, and $\ell' > \frac{n}{\sigma_1}$. Therefore, Hölder's inequality and Theorem 5 imply

$$\int_{B_R(0)^C} \frac{v(y)^q}{|y|^{\sigma_1}} \, dy \le \Big(\int_{B_R(0)^C} \frac{1}{|y|^{\sigma_1 \frac{\ell}{\ell-1}}} \, dy \Big)^{\frac{\ell-1}{\ell}} \Big(\int_{B_R(0)^C} v(y)^{\ell q} \, dy \Big)^{1/\ell}$$

$$\le C \Big(\int_R^\infty t^{n-\sigma_1 \ell'} \frac{dt}{t} \Big)^{\frac{1}{\ell'}} \Big(\int_{B_R(0)^C} v(y)^{\ell q} \, dy \Big)^{1/\ell}$$

$$\le C \|v\|_{\ell q}^q < \infty.$$

(2.) Assume $n - \alpha > p(n - \alpha) - (\alpha - \sigma_2)$. For small $\varepsilon > 0$, take $\ell = \frac{n}{n - \sigma_1 + \varepsilon}$ and $\ell' = \frac{n}{\sigma_1 - \varepsilon}$ so that $\frac{1}{\ell} + \frac{1}{\ell'} = 1$. From the non-subcritical condition (1.8) and since $pq > 1$, we get

$$\frac{q(n - \sigma_2) + (n - \sigma_1)}{q(1 + p)} = \frac{n - \sigma_1}{q(1 + p)} + \frac{n - \sigma_2}{1 + p} < \frac{n - \sigma_1}{1 + q} + \frac{n - \sigma_2}{1 + p} \leq n - \alpha.$$
(4.2)

Thus, $\frac{n - \sigma_1}{q(1 + p)} < n - \alpha - \frac{n - \sigma_2}{1 + p} \implies \frac{n - \sigma_1}{q} < p(n - \alpha) - (\alpha - \sigma_2)$. This yields $\frac{n - \sigma_1 + \varepsilon}{nq} < \frac{p(n - \alpha) - (\alpha - \sigma_2)}{n}$ for a sufficiently small ε, which implies

$$\frac{1}{\ell q} < \frac{p(n - \alpha) - (\alpha - \sigma_2)}{n} \quad \text{and} \quad \ell' > \frac{n}{\sigma_1}.$$

Hence, Hölder's inequality and Theorem 5 imply

$$\int_{B_R(0)^C} \frac{v(y)^q}{|y|^{\sigma_1}} \, dy \leq C \|v\|_{lq}^q < \infty.$$

(ii) For fixed $R > 0$, write

$$\int_{\mathbb{R}^n} \frac{|x|^{n - \alpha} v(y)^q}{|x - y|^{n - \alpha} |y|^{\sigma_1}} \, dy = \int_{B_R(0)} \frac{|x|^{n - \alpha} v(y)^q}{|x - y|^{n - \alpha} |y|^{\sigma_1}} \, dy$$

$$+ \int_{\{\mathbb{R}^n \backslash B_R(0)\} \backslash B_{|x|/2}(x)} \frac{|x|^{n - \alpha} v(y)^q}{|x - y|^{n - \alpha} |y|^{\sigma_1}} \, dy + \int_{B_{|x|/2}(x)} \frac{|x|^{n - \alpha} v(y)^q}{|x - y|^{n - \alpha} |y|^{\sigma_1}} \, dy$$

$$\doteq J_1 + J_2 + J_3.$$

Set

$$J_1' \doteq \int_{B_R(0)} \frac{v(y)^q}{|y|^{\sigma_1}} \left(\frac{|x|^{n - \alpha}}{|x - y|^{n - \alpha}} - 1 \right) dy.$$

For $y \in B_R(0)$ and large $|x|$,

$$\frac{v(y)^q}{|y|^{\sigma_1}} \left| \frac{|x|^{n - \alpha}}{|x - y|^{n - \alpha}} - 1 \right| \leq C \frac{v(y)^q}{|y|^{\sigma_1}} \in L^1(\mathbb{R}^n)$$

as a result of part (i). By virtue of the Lebesgue dominated convergence theorem, $|J_1'| \longrightarrow 0$ as $|x| \longrightarrow \infty$, which implies

$$\lim_{R \longrightarrow \infty} \lim_{|x| \longrightarrow \infty} J_1 = A_0. \tag{4.3}$$

Next, notice that if $y \in \{\mathbb{R}^n \backslash B_R(0)\} \backslash B_{|x|/2}(x)$, then $|x - y| \geq |x|/2$. Therefore,

$$J_2 \leq C \int_{\mathbb{R}^n \backslash B_R(0)} \frac{v(y)^q}{|y|^{\sigma_1}} \, dy \longrightarrow 0 \quad \text{as } R \longrightarrow \infty. \tag{4.4}$$

Set
$$J_3' \doteq \frac{J_3}{|x|^{n-\alpha}} = \int_{B_{|x|/2}(x)} \frac{v(y)^q}{|x-y|^{n-\alpha}|y|^{\sigma_1}} \, dy.$$

In view of Theorem 7, v is radially symmetric and decreasing about the origin. Therefore,

$$J_3' \le v(|x|/2)^q \int_{B_{|x|/2}(x)} \frac{dy}{|x-y|^{n-\alpha}|y|^{\sigma_1}} \le Cv(|x|/2)^q |x|^{\alpha-\sigma_1}. \qquad (4.5)$$

By Theorem 5, $v \in L^s(\mathbb{R}^n)$ such that

$$\frac{1}{s} = \min\left\{\frac{n-\alpha}{n}, \frac{p(n-\alpha)-(\alpha-\sigma_2)}{n}\right\} - \frac{\varepsilon}{n} \quad \text{for sufficiently small } \varepsilon > 0.$$

Then, combining this with the decreasing property of v yields

$$v(|x|/2)^s |x|^n \le C \int_{B_{|x|/2}(0) \setminus B_{|x|/4}(0)} v(y)^s \, dy < \infty. \qquad (4.6)$$

We claim that
$$|x|^{n-\alpha} J_3' = \mathrm{o}(1) \quad \text{as } |x| \longrightarrow \infty. \qquad (4.7)$$

To do so, we consider two cases.

Case 1. Let $n - \alpha \le p(n-\alpha) - (\alpha - \sigma_2)$. Then (4.6) implies that

$$v(|x|/2)^q |x|^{q(n-\alpha-\varepsilon)} \le C,$$

which when combined with (4.5), yields

$$|x|^{q(n-\alpha-\varepsilon)-(\alpha-\sigma_1)} J_{3'} \le Cv(|x|/2)^q |x|^{q(n-\alpha-\varepsilon)} \le C.$$

Recall that $q \ge \frac{n+\alpha-2\sigma_1}{n-\alpha}$, which implies that $q(n-\alpha)-(\alpha-\sigma_1) \ge n-\sigma_1 > n - \alpha$. Thus, by choosing ε suitably small and sending $|x| \longrightarrow \infty$ in the previous estimate after the appropriate calculations, we obtain (4.7).

Case 2. Let $n - \alpha > p(n-\alpha) - (\alpha - \sigma_2)$. Then (4.6) implies

$$v(|x|/2)^q |x|^{q(p(n-\alpha)-(\alpha-\sigma_2))} \le C,$$

which, when combined with (4.5), gives us

$$|x|^{q[p(n-\alpha)-(\alpha-\sigma_2)]-(\alpha-\sigma_1)} J_{3'} \le C.$$

It is easy to check that (4.2) implies that

$$q[p(n-\alpha)-(\alpha-\sigma_2)]-(\alpha-\sigma_1) > n - \alpha.$$

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Assertion (4.7) follows by sending $|x| \longrightarrow \infty$ in the last estimate after the appropriate calculations.

Notice that (4.7) implies that

$$\lim_{|x| \longrightarrow \infty} J_3 = 0. \tag{4.8}$$

Hence, (4.3),(4.4) and (4.8) imply

$$\lim_{|x| \longrightarrow \infty} |x|^{n-\alpha} u(x) = A_0,$$

and this completes the proof of the proposition. $\qquad\qquad\square$

4.2 Fast decay rates for $v(x)$

Proposition 4. *If $p(n - \alpha) + \sigma_2 > n$, then*

(i) $A_1 \doteq \displaystyle\int_{\mathbb{R}^n} \frac{u(y)^p}{|x|^{\sigma_2}} \, dy < \infty;$

(ii) $\displaystyle\lim_{|x| \longrightarrow \infty} |x|^{n-\alpha} v(x) = A_1.$

Proof. Since $p > \frac{n - \sigma_2}{n - \alpha}$, (i) follows from Theorem 5 and Hölder's inequality similar to the proof of Proposition 3(i).

To prove (ii), write

$$|x|^{n-\alpha} v(x) = \int_{B_R(0)} \frac{|x|^{n-\alpha} u(y)^p}{|x - y|^{n-\alpha} |y|^{\sigma_2}} \, dy$$

$$+ \int_{\mathbb{R}^n \backslash B_R(0)} \frac{|x|^{n-\alpha} [A_0 + o(1)]^p}{|x - y|^{n-\alpha} |y|^{p(n-\alpha)+\sigma_2}} \, dy \doteq J_4 + J_5.$$

For a large $R > 0$, if $y \in B_R(0)$, then $\lim_{|x| \longrightarrow \infty} \frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}} = 1$, and the Lebesgue dominated convergence theorem implies that

$$\lim_{R \longrightarrow \infty} \lim_{|x| \longrightarrow \infty} J_4 = \lim_{R \longrightarrow \infty} \lim_{|x| \longrightarrow \infty} \int_{B_R(0)} \frac{|x|^{n-\alpha} u(y)^p}{|x - y|^{n-\alpha} |y|^{\sigma_2}} \, dy = A_1. \tag{4.9}$$

Likewise, since $p(n - \alpha) + \sigma_2 > n$,

$$J_5 \leq C \int_R^\infty t^{n-p(n-\alpha)-\sigma_2} \frac{dt}{t} = o(1) \ \text{ as } \ R \longrightarrow \infty \ \text{ for suitably large } |x|.$$

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Particularly,

$$\int_{\mathbb{R}^n \setminus B_R(0)} \frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}|y|^{p(n-\alpha)+\sigma_2}} \, dy = o(1) \ \text{ as } \ |x| \longrightarrow \infty, \ R \longrightarrow \infty.$$

Hence, assertion (ii) follows from this and (4.9). $\qquad\square$

Proposition 5. *If $p(n-\alpha) + \sigma_2 = n$, then*

$$\lim_{|x| \longrightarrow \infty} \frac{|x|^{n-\alpha}}{\ln|x|} v(x) = A_0^p |S^{n-1}|,$$

where $|S^{n-1}|$ is the surface area of the $(n-1)$-dimensional unit sphere.

Proof. From (4.1), we have for large $R > 0$,

$$\frac{|x|^{n-\alpha}}{\ln|x|} v(x) = \frac{1}{\ln|x|} \int_{B_R(0)} \frac{|x|^{n-\alpha} u(y)^p}{|x-y|^{n-\alpha}|y|^{\sigma_2}} \, dy$$
$$+ \frac{(A_0 + o(1))^p}{\ln|x|} \int_{\mathbb{R}^n \setminus B_R(0)} \frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}|y|^n} \, dy.$$

Indeed, (4.9) implies that

$$\frac{1}{\ln|x|} \int_{B_R(0)} \frac{|x|^{n-\alpha} u(y)^p}{|x-y|^{n-\alpha}|y|^{\sigma_2}} \, dy = o(1) \ \text{ as } \ |x| \longrightarrow \infty.$$

Thus, it only remains to show that

$$\frac{1}{\ln|x|} \int_{\mathbb{R}^n \setminus B_R(0)} \frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}|y|^n} \, dy \longrightarrow |S^{n-1}| \ \text{ as } \ |x| \longrightarrow \infty. \qquad (4.10)$$

Indeed, for large $R > 0$ and $c \in (0, 1/2)$, polar coordinates and a change of variables give us

$$\frac{1}{\ln|x|} \int_{\mathbb{R}^n \setminus B_R(0)} \frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}|y|^n} \, dy = \frac{1}{\ln|x|} \int_{\frac{R}{|x|}}^{c} \int_{S^{n-1}} \frac{1}{r|e-rw|^{n-\alpha}} \, dS_w dr$$
$$+ \frac{1}{\ln|x|} \int_{\mathbb{R}^n \setminus B_c(0)} \frac{1}{|z|^n |e-z|^{n-\alpha}} \, dz,$$

where e is a unit vector in $\mathbb{R}^n$. Clearly, the integral in the second term can be bounded above by a positive constant depending only on c, since

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$n - \alpha < n$ for z near e and $n - \alpha + n > n$ near infinity. Thus, we deduce that

$$\frac{1}{\ln |x|} \int_{\mathbb{R}^n \setminus B_c(0)} \frac{1}{|z|^n |e - z|^{n-\alpha}} \, dz = o(1) \quad \text{as } |x| \longrightarrow \infty.$$

For $r \in (0, c)$, it is also clear that $1 - c \leq |e - rw| \leq 1 + c$. Therefore, there exists $\theta \in (-1, 1)$ such that $|e - rw| = 1 + \theta c$, which leads to

$$\frac{1}{\ln |x|} \int_{\frac{R}{|x|}}^{c} \int_{S^{n-1}} \frac{1}{r |e - rw|^{n-\alpha}} \, dw dr = \frac{|S^{n-1}|}{(1 + \theta c)^{n-\alpha} \ln |x|} (\ln c - \ln R + \ln |x|)$$

$$\longrightarrow \frac{|S^{n-1}|}{(1 + \theta c)^{n-\alpha}} \quad \text{as } |x| \longrightarrow \infty.$$

Hence, by sending $c \longrightarrow 0$, we obtain (4.10) and this concludes the proof. $\square$

Proposition 6. *If $p(n - \alpha) + \sigma_2 < n$, then*

$$A_2 \doteq A_0^p \int_{\mathbb{R}^n} \frac{dz}{|z|^{p(n-\alpha)+\sigma_2} |e - z|^{n-\alpha}} < \infty.$$

Moreover,

$$\lim_{|x| \longrightarrow \infty} |x|^{p(n-\alpha)-(\alpha-\sigma_2)} v(x) = A_2.$$

Proof. According to the non-subcritical condition, we have that $\frac{n-\sigma_2}{1+p} < n - \alpha$. Therefore, the integrand in A_2 decays with the following rates: $(1+p)(n-\alpha) + \sigma_2 > n$ near infinity; $n - \alpha < n$ near e; and $p(n-\alpha) + \sigma_2 < n$ near the origin. Thus, we conclude that $A_2 < \infty$.

For large $R > 0$, we use (4.1) to write

$$|x|^{p(n-\alpha)-(\alpha-\sigma_2)} v(x) = |x|^{p(n-\alpha)+\sigma_2-n} \int_{B_R(0)} \frac{|x|^{n-\alpha} u(y)^p}{|x - y|^{n-\alpha} |y|^{\sigma_2}} \, dy$$

$$+ (A_0 + o(1))^p \int_{\mathbb{R}^n \setminus B_R(0)} \frac{|x|^{(p+1)(n-\alpha)+\sigma_2-n}}{|x - y|^{n-\alpha} |y|^{p(n-\alpha)+\sigma_2}} \, dy. \quad (4.11)$$

Indeed, if $y \in B_R(0)$, then

$$|x|^{p(n-\alpha)+\sigma_2-n} \int_{B_R(0)} \frac{|x|^{n-\alpha} u(y)^p}{|x - y|^{n-\alpha} |y|^{\sigma_2}} \, dy \leq C |x|^{p(n-\alpha)+\sigma_2-n} \longrightarrow 0$$

as $|x| \longrightarrow \infty$. Likewise, as $|x| \longrightarrow \infty$

$$\int_{\mathbb{R}^n \setminus B_R(0)} \frac{|x|^{(p+1)(n-\alpha)+\sigma_2-n}}{|x - y|^{n-\alpha} |y|^{p(n-\alpha)+\sigma_2}} \, dy = \int_{\mathbb{R}^n \setminus B_{R/|x|}(0)} \frac{dz}{|z|^{p(n-\alpha)+\sigma_2} |e - z|^{n-\alpha}}$$

$$\longrightarrow \frac{A_2}{A_0^p}.$$

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Inserting these calculations into (4.11) leads to the desired result. □

4.3   Characterization of integrable solutions

Proposition 7. *Let u, v be positive solutions of (1.1) satisfying (1.7). If u, v are bounded and decay with the fast rates as $|x| \longrightarrow \infty$, then u, v are integrable solutions.*

Proof. Suppose u, v are bounded and decay with the fast rates. From (1.7), it is clear that $(n - \alpha)r_0 > n$. Thus,

$$\int_{\mathbb{R}^n} u(x)^{r_0} \, dx \leq C + \int_{\mathbb{R}^n \setminus B_R(0)} u(x)^{r_0} \, dx \leq C_1 + C_2 \int_R^\infty t^{n - (n - \alpha) r_0} \frac{dt}{t} < \infty.$$

Similarly, $\int_{\mathbb{R}^n} v(x)^{s_0} \, dx < \infty$ if v decays with rate $|x|^{-(n - \alpha)}$. If v decays with the rate $|x|^{-(n - \alpha)} \ln |x|$, then we can find a suitably large $R > 0$ and small $\varepsilon > 0$ for which $(\ln |x|)^{s_0} \leq |x|^\varepsilon$ for $|x| > R$. Then, we also get $n - (n - \alpha) s_0 + \varepsilon < 0$ provided ε is sufficiently small and this implies

$$\int_{\mathbb{R}^n} v(x)^{s_0} \, dx \leq C_1 + C_2 \int_R^\infty t^{n - (n - \alpha) s_0 + \varepsilon} \frac{dt}{t} < \infty.$$

If v decays with the rate $|x|^{-(p(n - \alpha) - (\alpha - \sigma_2))}$, then (1.7) implies that $q_0 < n - \alpha$, which further yields $pq_0 - \alpha + \sigma_2 < p(n - \alpha) - (\alpha - \sigma_2)$. From this we deduce that $n - (p(n - \alpha) - (\alpha - \sigma_2)) s_0 < 0$. Therefore,

$$\int_{\mathbb{R}^n} v(x)^{s_0} \, dx \leq C_1 + C_2 \int_R^\infty t^{n - (p(n - \alpha) - (\alpha - \sigma_2)) s_0} \frac{dt}{t} < \infty.$$

In any case, we conclude that $(u, v) \in L^{r_0}(\mathbb{R}^n) \times L^{s_0}(\mathbb{R}^n)$. □

Proof of Theorem 3. Propositions 3–7 show u, v are positive integrable solutions if and only if they are bounded and decay with the fast rates as $|x| \longrightarrow \infty$. Lastly, it remains to show that (1.1) does not admit any positive integrable solution in the supercritical case. To prove this, assume u and v are positive integrable solutions. Then, we can apply similar arguments found in the proof of Proposition 3 to show that

$$\int_{\mathbb{R}^n} \frac{u(x)^{p+1}}{|x|^{\sigma_2}} \, dx = \int_{\mathbb{R}^n} \frac{v(x)^{q+1}}{|x|^{\sigma_1}} \, dx < \infty.$$

Then, as in the proof of Theorem 2, we can deduce the same Pohozaev type identity to arrive at

$$\left\{ \frac{n - \sigma_1}{1 + q} + \frac{n - \sigma_2}{1 + p} - (n - \alpha) \right\} \int_{\mathbb{R}^n} \frac{v(x)^{q+1}}{|x|^{\sigma_1}} \, dx = 0,$$

but this contradicts the supercritical condition. This completes the proof of the theorem. □

5 Asymptotic properties of non-integrable solutions

In this section, we assume u, v are bounded positive solutions of system (1.1).

Proposition 8. *Let $\theta_1 < q_0$ and $\theta_2 < p_0$. Then there does not exist a positive constant c such that either*

$$u(x) \geq c(1 + |x|)^{-\theta_1} \ \ or \ \ v(x) \geq c(1 + |x|)^{-\theta_2}.$$

Proof. Assume that there exists such a $c > 0$ in which,

$$u(x) \geq c(1 + |x|)^{-\theta_1} \ \ \text{where} \ \theta_1 < q_0.$$

Then there holds for large x,

$$v(x) \geq \int_{B_{|x|/2}(x)} \frac{u(y)^p}{|x - y|^{n-\alpha}|y|^{\sigma_2}} \, dy \geq c(1 + |x|)^{-a_1},$$

where $b_0 = \theta_1$ and $a_1 = pb_0 - \alpha + \sigma_2$. Thus, inserting this into the first integral equation yields

$$u(x) \geq \int_{B_{|x|/2}(x)} \frac{v(y)^q}{|x - y|^{n-\alpha}|y|^{\sigma_1}} \, dy \geq c(1 + |x|)^{-b_1},$$

where $b_1 = qa_1 - \alpha + \sigma_1$. By inductively repeating this argument, we arrive at

$$v(x) \geq c(1 + |x|)^{-a_j} \ \ \text{and} \ \ u(x) \geq c(1 + |x|)^{-b_j},$$

where

$$a_{j+1} = pb_j - \alpha + \sigma_2 \ \ \text{and} \ \ b_j = qa_j - \alpha + \sigma_1 \ \ \text{for} \ \ j = 1, 2, 3, \ldots.$$

A simple calculation yields

$$
\begin{aligned}
b_k &= q a_k - \alpha + \sigma_1 = q(p b_{k-1} - \alpha + \sigma_2) - \alpha + \sigma_1 \\
&= pq b_{k-1} - (\alpha(1+q) - (\sigma_1 + \sigma_2 q)) \\
&= (pq)^2 b_{k-2} - (\alpha(1+q) - (\sigma_1 + \sigma_2 q))(1 + pq) \\
&\ \ \vdots \\
&= (pq)^k b_0 - (\alpha(1+q) - (\sigma_1 + \sigma_2 q))(1 + pq + (pq)^2 + \ldots + (pq)^{k-1}) \\
&= (pq)^k b_0 - (\alpha(1+q) - (\sigma_1 + \sigma_2 q)) \frac{(pq)^k - 1}{pq - 1} = (pq)^k (b_0 - q_0) + q_0.
\end{aligned}
$$

Since $pq > 1$ and $b_0 = \theta_1 < q_0$, we can find a sufficiently large k_0 such that $b_{k_0} < 0$, but then this implies that for a suitable choice of $R > 0$,

$$
v(x) \geq c \int_{\mathbb{R}^n \setminus B_R(0)} \frac{u(y)^p}{|x-y|^{n-\alpha} |y|^{\sigma_2}} \, dy \geq c \int_{\mathbb{R}^n \setminus B_R(0)} \frac{|y|^{-p b_{k_0}}}{|x-y|^{n-\alpha} |y|^{\sigma_2}} \, dy
$$

$$
\geq c \int_R^\infty t^{\alpha - \sigma_2 - p b_{k_0}} \frac{dt}{t} = \infty.
$$

Hence $v(x) = \infty$, which is impossible. Similarly, if there exists a $c > 0$ such that

$$
v(x) \geq c(1 + |x|)^{-\theta_2} \quad \text{where } \theta_2 < p_0,
$$

then we can apply the same iteration argument to conclude $u(x) = \infty$ for large x and this completes the proof. $\qquad \square$

Proposition 9. *There hold the following.*

(i) *Let $\theta_3 > q_0$ and $\theta_4 > p_0$. If u, v are not integrable solutions, then there does not exist a positive constant C such that either*

$$
u(x) \leq C(1 + |x|)^{-\theta_3} \quad \text{or} \quad v(x) \leq C(1 + |x|)^{-\theta_4}.
$$

(ii) *If u, v are not integrable solutions but are decaying solutions, i.e.*

$$
u(x) \simeq |x|^{-\theta_1} \quad \text{and} \quad v(x) \simeq |x|^{-\theta_2}
$$

for some $\theta_1, \theta_2 > 0$, then they necessarily have the slow rates $\theta_1 = q_0$ and $\theta_2 = p_0$.

Proof. (i) On the contrary, assume there exists a $C > 0$ such that $u(x) \leq C(1 + |x|)^{-\theta_3}$. Then $n - r_0\theta_3 < 0$ and we calculate that

$$\int_{\mathbb{R}^n} u(x)^{r_0} \, dx = \int_{B_R(0)} u(x)^{r_0} \, dx + \int_{\mathbb{R}^n \setminus B_R(0)} u(x)^{r_0} \, dx$$
$$\leq C_1 + C_2 \int_R^\infty t^{n-r_0\theta_3} \frac{dt}{t} < \infty,$$

which contradicts the assumption that u, v are not integrable solutions. Likewise, if there exists a $C > 0$ such that $v(x) \leq C(1 + |x|)^{-\theta_4}$, a similar argument shows that $v \in L^{s_0}(\mathbb{R}^n)$, which is a contradiction.

(ii) Now suppose that u, v are non-integrable solutions but are decaying solutions. Then part (i) and Proposition 8 clearly imply that u, v decay with the slow rates as $|x| \longrightarrow \infty$. □

Proof of Theorem 4. Part (i) of the theorem follows immediately from Proposition 8 and parts (ii) and (iii) follow from Proposition 9. □

References

[1] F. Arthur, X. Yan, and M. Zhao. A Liouville-type theorem for higher order elliptic systems. *Discrete Contin. Dyn. Syst.*, 34(9):3317–3339, 2014.

[2] M. Badiale and G. Tarantello. A Hardy–Sobolev inequality with applications to a nonlinear elliptic equation arising in astrophysics. *Arch. Ration. Mech. Anal.*, 163:259–293, 2002.

[3] L. Caffarelli, R. Kohn, and L. Nirenberg. First order interpolation inequalities with weights. *Compos. Math.*, 53(3):259–275, 1984.

[4] G. Caristi, L. D'Ambrosio, and E. Mitidieri. Representation formulae for solutions to some classes of higher order systems and related Liouville theorems. *Milan J. Math.*, 76(1):27–67, 2008.

[5] F. Catrina and Z. Wang. On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions. *Comm. Pure Appl. Math.*, 54(2):229–258, 2001.

[6] W. Chen and C. Li. Regularity of solutions for a system of integral equations. *Commun. Pure Appl. Anal.*, 4:1–8, 2005.

[7] W. Chen and C. Li. An integral system and the Lane-Emden conjecture. *Discrete Contin. Dyn. Syst.*, 4(24):1167–1184, 2009.

[8] W. Chen and C. Li. Super polyharmonic property of solutions for PDE systems and its applications. *Commun. Pure Appl. Anal.*, 12(6):2497–2514, 2013.

[9] W. Chen, C. Li, and B. Ou. Classification of solutions for a system of integral equations. *Comm. in Partial Differential Equations*, 30:59–65, 2005.

[10] W. Chen, C. Li, and B. Ou. Qualitative properties of solutions for an integral equation. *Discrete Contin. Dyn. Syst.*, 12:347–354, 2005.

[11] W. Chen, C. Li, and B. Ou. Classification of solutions for an integral equation. *Comm. Pure Appl. Math.*, 59:330–343, 2006.

[12] K. S. Chou and C. W. Chu. On the best constant for a weighted Sobolev–Hardy inequality. *J. Lond. Math. Soc.*, 2:137–151, 1993.

[13] L. D'Ambrosio and E. Mitidieri. Hardy-Littlewood-Sobolev systems and related Liouville theorems. *Discrete Contin. Dyn. Syst. Ser. S*, 7(4):653–671, 2014.

[14] F. Hang. On the integral systems related to Hardy-Littlewood-Sobolev inequality. *Math. Res. Lett.*, 14(3):373–383, 2007.

[15] M. Fazly. Liouville theorems for the polyharmonic Hénon–Lane–Emden system. *Methods Appl. Anal.*, 21(2):265–282, 2014.

[16] M. Fazly and N. Ghoussoub. On the Hénon-Lane-Emden conjecture. *Discrete Contin. Dyn. Syst.*, 34(6):2513–2533, 2014.

[17] B. Gidas and J. Spruck. A priori bounds for positive solutions of nonlinear elliptic equations. *Comm. Partial Differential Equations*, 6(8):883–901, 1981.

[18] C. Jin and C. Li. Qualitative analysis of some systems of integral equations. *Calc. Var. Partial Differential Equations*, 26:447–457, 2005.

[19] Y. Lei. Asymptotic properties of positive solutions of the Hardy-Sobolev type equations. *J. Differential Equations*, 254:1774–1799, 2013.

[20] Y. Lei and C. Li. Decay properties of the Hardy-Littlewood-Sobolev systems of the Lane-Emden type, preprint, *http://arXiv:1302.5567, 2013.*

[21] Y. Lei and C. Li. Sharp criteria of Liouville type for some nonlinear systems, preprint, *http://arXiv:1301.6235, 2013.*

[22] Y. Lei, C. Li, and C. Ma. Asymptotic radial symmetry and growth estimates of positive solutions to weighted Hardy–Littlewood–Sobolev system of integral equations. *Calc. Var. Partial Differential Equations*, 45(1-2):43–61, 2012.

[23] Y. Lei and C. Ma. Asymptotic behavior for solutions of some integral equations. *Commun. Pure Appl. Anal.*, 10:193–207, 2011.

[24] C. Li. A degree theory approach for the shooting method, preprint, *http://arXiv:1301.6232, 2013.*

[25] C. Li and J. Lim. The singularity analysis of solutions to some integral equations. *Commun. Pure Appl. Anal.*, 6(2):453–464, 2007.

[26] Y. Li. Asymptotic behavior of positive solutions of equation $\Delta u + K(x)u^p = 0$ in $\mathbb{R}^n$. *J. Differential Equations*, 95:304–330, 1992.

[27] Y. Li and W. M. Ni. On conformal scalar curvature equations in $\mathbb{R}^n$. *Duke Math. J.*, 57:895–924, 1988.

[28] Y. Y. Li. Remark on some conformally invariant integral equations: the method of moving spheres. *J. Eur. Math. Soc.*, 6(2):153–180, 2004.

[29] E. Lieb. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. of Math.*, 118:349–374, 1983.

[30] J. Liu, Y. Guo, and Y. Zhang. Existence of positive entire solutions for polyharmonic equations and systems. *J. Partial Differ. Equ.*, 19(3):256–270, 2006.

[31] J. Liu, Y. Guo, and Y. Zhang. Liouville-type theorems for polyharmonic systems in R^N. *J. Differential Equations*, 225(2):685–709, 2006.

[32] G. Lu and J. Zhu. Symmetry and regularity of extremals of an integral equation related to the Hardy–Sobolev inequality. *Calc. Var. Partial Differential Equations*, 42(3-4):563–577, 2011.

[33] E. Mitidieri. Nonexistence of positive solutions of semilinear elliptic systems in R^N. *Differ. Integral Equations*, 9:465–480, 1996.

[34] Q. H. Phan. Liouville-type theorems and bounds of solutions for Hardy-Hénon systems. *Adv. Differential Equations*, 17(7-8):605–634, 2012.

[35] P. Poláčik, P. Quittner, and P. Souplet. Singularity and decay estimates in superlinear problems via Liouville-type theorems, I: Elliptic equations and systems. *Duke Math. J.*, 139(3):555–579, 2007.

[36] J. Serrin and H. Zou. Non-existence of positive solutions of Lane-Emden systems. *Differ. Integral Equations*, 9(4):635–653, 1996.

[37] P. Souplet. The proof of the Lane–Emden conjecture in four space dimensions. *Adv. Math.*, 221(5):1409–1427, 2009.

[38] E. B. Stein and G. Weiss. Fractional integrals on n-dimensional Euclidean space. *J. Math. Mech.*, 7(4):503–514, 1958.

[39] J. Villavert. Shooting with degree theory: Analysis of some weighted polyharmonic systems. *J. Differential Equations*, 257(4):1148–1167, 2014.

[40] J. Villavert. Sharp existence criteria for positive solutions of Hardy–Sobolev type systems. *Commun. Pure Appl. Anal.*, 14(2):493–515, 2015.

[41] J. Wei and X. Xu. Classification of solutions of higher order conformally invariant equations. *Math. Ann.*, 313:207–228, 1999.

[42] Y. Zhao and Y. Lei. Asymptotic behavior of positive solutions of a nonlinear integral system. *Nonlinear Anal.*, 75(4):1989–1999, 2012.